

IV. "On the Motion of a Sphere in a Viscous Liquid." By A. B. BASSET, M.A. Communicated by LORD RAYLEIGH, D.C.L., Sec. R.S. Received November 10, 1887.

(Abstract.)

The determination of the small oscillations and steady motion of a sphere which is immersed in a viscous liquid, and which is moving in a straight line, was first effected by Professor Stokes in his well-known memoir "On the Effect of the Internal Friction of Fluids on the Motion of Pendulums;"\* and in the appendix he also determines the steady motion of a sphere which is rotating about a fixed diameter. The same subject has also been subsequently considered by Helmholtz and other German writers; but, so far as I have been able to discover, very little appears to have been effected with respect to the solution of problems in which a solid body is set in motion in a viscous liquid in any given manner, and then left to itself.

In the present paper I have endeavoured to determine the motion of a sphere which is projected vertically upwards or downwards with given velocity, and allowed to ascend or descend under the action of gravity (or any constant force), and which is surrounded by a viscous liquid of unlimited extent, which is initially at rest excepting so far as it is disturbed by the initial motion of the sphere.

In solving this problem, mathematical difficulties have compelled me to neglect the squares and products of velocities, and quantities depending thereon, which involves the assumption that the velocity of the sphere is always small throughout the motion; and I have also assumed that no slipping takes place at the surface of the sphere. The problem is thus reduced to obtaining a suitable solution of the differential equation

$$D\left(D - \frac{1}{\mu} \frac{d}{dt}\right)\psi,$$

$$\text{where} \quad D = \frac{d^2}{dr^2} + \frac{\sin \theta}{r} \frac{d}{d\theta} \left( \operatorname{cosec} \theta \frac{d}{d\theta} \right).$$

$\psi$  is Stokes's current function, and  $\mu$  is the kinematic coefficient of viscosity. The required solution is obtained in the form of a definite integral by a method similar to that employed by Fourier in solving analogous problems in the conduction of heat; the resistance experienced by the sphere is then calculated, and the equation of motion written down and integrated by successive approximation on the supposition that  $\mu$  is a small quantity. The values of the acceleration and velocity of the sphere to a third approximation are found to be

\* 'Cambridge Phil. Soc. Trans.,' vol. 9.

$$\dot{v} = f\epsilon^{-\lambda t} - V\lambda\epsilon^{-\lambda t} - fka\sqrt{\frac{\mu}{\pi}}\left\{\left(\frac{1}{2} - \lambda t\right)\phi(t) + \sqrt{t}\right\} \\ + f k^2 a^2 \mu t \epsilon^{-\lambda t} \left(1 - \frac{1}{2}\lambda t\right),$$

$$v = \frac{f}{\lambda}(1 - \epsilon^{\lambda t}) + V\epsilon^{-\lambda t} - fka\sqrt{\frac{\mu}{\pi}}\left\{\left(t + \frac{1}{2\lambda}\right)\phi(t) - \frac{\sqrt{t}}{\lambda}\right\} \\ + \frac{1}{2} f k^2 a^2 \mu t^2 \epsilon^{-\lambda t},$$

where

$$f = \frac{(\sigma - \rho)g}{\sigma + \frac{1}{2}\rho}, \quad k = \frac{9\rho}{a^2(2\sigma + \rho)}, \quad \lambda = k\mu, \quad \phi(t) = \int_0^t \epsilon^{-\lambda\tau}(t - \tau)^{-\frac{1}{2}} d\tau,$$

$\rho$  being the density of the liquid,  $\sigma$  that of the sphere, and  $a$  its radius.

It thus appears that after a very long time has elapsed, the acceleration will vanish and the motion will become steady. The terminal velocity of the sphere is  $f\lambda^{-1}$ , which is seen to agree with Professor Stokes's result.

If the sphere were projected with velocity  $V$ , and compelled by means of frictionless constraint to move in a horizontal straight line, the values of the acceleration and velocity would be obtained from the preceding formulæ by expunging the terms  $f\epsilon^{-\lambda t}$ ,  $f\lambda^{-1}(1 - \epsilon^{-\lambda t})$ , in the expressions for  $\dot{v}$  and  $v$  respectively, and then changing  $f$  into  $-V\lambda$ .

The preceding results can only be regarded as a somewhat rough representation of the actual motion, for (i) the square of the velocity has been neglected; (ii) no account has been taken of the possibility of hollow spaces being formed in the liquid; (iii) if the velocity of the sphere became large, the amount of heat developed would be sufficient to vaporise the liquid in the immediate neighbourhood of the sphere, and the circumstances of the problem would be materially changed.

In the latter part of the paper I have considered the problem of a sphere, surrounded by a viscous liquid, which is set in rotation with given angular velocity,  $\Omega$ , about a fixed diameter, and similar results are obtained. To a first approximation the angular velocity is equal to  $\Omega\epsilon^{-\lambda t}$ , where  $\lambda$  is a positive constant, which shows that the motion ultimately dies away.